

Almost-Everywhere Superiority for Quantum Computing*

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Abstract

Simon [Sim97] as extended by Brassard and Høyer [BH97] shows that there are problems, relative to black-box functions, on which quantum machines are exponentially faster than each classical machine infinitely often. The present paper shows that there are problems, relative to black-box functions, on which quantum machines are exponentially faster than each classical machine almost everywhere.

1 Overview

Work of Simon [Sim97], as extended by Brassard and Høyer [BH97], is often cited as key evidence for the potential superiority of quantum computation over classical computation. Their work shows that for computation relative to a black-box function (also sometimes referred to as a promise function) there are problems on which polynomial-time quantum computation is infinitely often exponentially faster than each deterministic—or even bounded-error probabilistic—classical computer solving the problem.

Note that this leaves open the possibility that for some classical computers solving the problem there are infinitely many inputs on which quantum computing is *not* interestingly faster on these problems, and indeed such is the case. In fact, quantum computing in Simon's construction is superior on only an exponentially small portion of inputs.

In contrast, the present paper obtains problems on which quantum computing is exponentially superior to classical computing *almost everywhere*. In particular, we show that for computation relative to a black-box function there are problems solved in polynomial-time by quantum computers but on which every deterministic—or even bounded-error probabilistic—classical computer solving the problem requires exponential time on all but a finite number of inputs.

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2 History and Discussion

2.1 Simon and Infinitely-Often Superiority for Quantum Computing

During the 1990s, tremendously exciting models of computation—such as quantum computing and DNA-computing—have been one strong focus of theoretical computer science research. Researchers dearly want to know whether these models, at least in certain settings, offer computational properties (most particularly, quick run-time) superior to what is offered by classical computational models.

Of course, even such an exciting model as quantum computing has limitations (see, e.g., the elegant lower-bound approach of Beals et al. [BBC⁺98]). However, let us here consider the highlights of what is known suggesting the superiority of quantum computing. The three most famous lines of work are those of Shor [Sho97], Grover ([Gro96], see also [BBHT98]), and Simon ([Sim97], see also [BH97]).

Grover shows that quantum computing can do certain search problems at a quadratically faster exponential speed than one intuitively would expect in classical computing. Shor shows that factoring (and other interesting problems) can be done in expected polynomial time in the quantum model. These are both undeniably impressive results. However, note that it is at least plausible that classical, deterministic computing can search through huge numbers of possibilities very quickly and can factor very quickly; for example, if $P = NP$, NP-like search problems¹ and factoring all are easily in P .

In contrast, Simon [Sim97] shows that for computing with respect to a black-box function there are problems for which quantum bounded-error computing provably *is* exponentially faster than classical deterministic computing or even classical bounded-error computing. Brassard and Høyer [BH97] improved the upper bound to obtain that for computing with respect to a black-box function there are problems for which quantum computing is exponentially faster than classical deterministic computing or even classical bounded-error computing, in particular, there are problems in QP such that each bounded-error classical Turing machine solving them requires exponential time on infinitely many inputs.

2.2 Limitations of Simon's Result

As described in the previous subsection, Simon-Brassard-Høyer show, for computing with respect to a black-box function, the infinitely-often exponential superiority of quantum computing over classical deterministic computing (and even over classical bounded-error computing), on a particular problem. Since we will always speak of computing with respect

¹To be fair to Grover, his result can plausibly be viewed instead as a black-box result. The key issue is whether the predicate, $C(S)$, that he uses should be viewed as some polynomial-time evaluation or as a black-box predicate. He does not have to address this issue (his motivating example, SAT, satisfies the former but a parenthetical remark in his paper suggests the latter), as his results are valid either way and as he is improving the upper bound rather than establishing any lower bounds. In any case, note that in contrast to Simon's and the present paper's exponential superiority results, Grover's algorithm beats the obvious brute-force deterministic algorithm by just a quadratic factor.

to a black-box function, we will henceforth stop mentioning that and take it to be implicit from context.

Are there any worries or limitations to Simon’s work? Simon (when one tightens his upper bound to QP via the work of Brassard-Høyer) gives an “infinitely often” result: a problem that is in QP but such that each classical bounded-error machine solving the problem takes exponential time on infinitely many inputs. However, “infinitely many” says no more than it seems to. In fact, for Simon’s problem, there are classical deterministic machines that solve the problem essentially instantly (i.e., in $n + 1$ steps on inputs of length n) on the vast majority of inputs—in fact, on all but one input of each length.

So, even though Simon proves infinitely-often superiority, in fact for his problem the superiority occurs only on an exponentially thin portion of inputs. In contrast, the present paper achieves exponential superiority on all sufficiently long inputs (so, for example, each classical machine for the problem will take subexponential time on at most a finite set of inputs).

This is well-motivated as one issue of broad importance in the area of quantum computing is to gain an understanding of exactly what potential quantum computers hold, i.e., what superiority over classical computers they offer.

3 Results: Almost-Everywhere Superiority for Quantum Computing

Let us start by explicitly stating where we will go. Recall that what Simon’s main theorem states (again, using here the Brassard-Høyer improvement of the upper bound to QP) is the following.

Theorem 3.1 (*[Sim97, Theorem 3.4] augmented by [BH97]*) *There is a constant $\epsilon > 0$ and a (function) oracle \mathcal{O} relative to which there is a language B in QP such that each bounded-error classical Turing machine accepting B requires time more than $2^{\epsilon n}$ on infinitely many inputs.*

What we will to prove is the following result, which extends the superiority from merely infinitely often to instead almost everywhere.

Theorem 3.2 *There is a constant $\epsilon > 0$ and a (function) oracle \mathcal{O} relative to which there is a problem B computable in quantum polynomial time such that each bounded-error classical Turing machine computing B requires time more than $2^{\epsilon n}$ on all but a finite number of inputs.*

It follows immediately that this problem also demonstrates the almost-everywhere superiority of quantum computation over deterministic computation, when computing relative to a black-box function.

Corollary 3.3 *There is a constant $\epsilon > 0$ and a (function) oracle \mathcal{O} relative to which there is a problem B computable in quantum polynomial time such that each deterministic classical*

Turing machine computing B requires time more than $2^{\epsilon n}$ on all but a finite number of inputs.

Some comments are in order regarding Theorem 3.2. First, we should mention that the computational task on which we prove almost everywhere exponential superiority for quantum computing is, in contrast with Simon’s task, a function rather than a language. Second, we should explicitly define what we mean by a probabilistic function.

Definition 3.4 *We say a function f is bounded-error Turing computable in time $T(n)$ (i.e., is in $\text{BPTIME}[T(n)]$) iff there is an $\epsilon > 0$ and a probabilistic Turing machine running in time $T(n)$ such that, on each $x \in \Sigma^*$,*

$$\text{Prob}(M(x) = f(x)) \geq 1/2 + \epsilon.$$

Finally, we review a bit about Simon’s result, as his result motivated our work, and as it is important to both point out why the obvious transformation of his result does not give the result we seek and as we should credit him for the connections between his construction and ours.

The key construction used by Simon is described in the statement of the following result.

Theorem 3.5 ([Sim97, Theorem 3.3]) *Let \mathcal{O} be a (function) oracle constructed as follows: for each n , a random n -bit string $s(n)$ and a random bit $b(n)$ are uniformly chosen from $\{0, 1\}^n$ and $\{0, 1\}$, respectively. If $b(n) = 0$, then the function $f_n : \{0, 1\}^n \rightarrow \{0, 1\}^n$ chosen for \mathcal{O} to compute on n -bit queries is a random function uniformly distributed over permutations on $\{0, 1\}^n$; otherwise it is a random function uniformly distributed over two-to-one functions such that $f_n(x) = f_n(x \oplus s(n))$ for all x , where \oplus denotes bitwise exclusive-or. Then any PTM (probabilistic Turing machine) that queries \mathcal{O} no more than $2^{n/4}$ times cannot correctly guess $b(n)$ with probability greater than $(1/2) + 2^{-n/2}$, over choices in the construction of \mathcal{O} .²*

Simon’s “test language” that, based on this oracle, gives one the lower-bound of Theorem 3.1 is quite simply the issue of testing the bit described above, that is, the test language that is in QP but on which bounded-error $2^{\epsilon n}$ -time classical Turing machines all err on infinitely many inputs is $\{1^n \mid b(n) = 1\}$.

It might be very tempting to exactly adopt the oracle \mathcal{O} of Simon, but using instead of his test language the new test language: $\widehat{L} = \{w \mid b(|w|) = 1\}$. This change attempts to “smear” the difficulty of 1^n onto all strings of length n , and even attempts to achieve the language analog of our desired result.

Unfortunately, this provably does not work. Why? A PTM can use the information in the input to (very rarely, but often enough) help it guess $s(n)$, in particular, certainly when the input happens to be the input that is $s(n)$.³

²The statement here is taken exactly from Simon. There are some informalities in Simon’s statement—the fact that what independence is assumed is not explicitly stated and that the case $s(n) = 0^n$ won’t give an (exactly-2)-to-1 function—and so we do our best to be as careful as possible in the proof of our main result.

³Just to be explicit here for absolute clarity, consider the PTM that on each input w does:

So, our construction takes a different tack. Intuitively speaking, the above problem should be removed if we increase the information content of the xor-bitmask well beyond that which input strings can give away. To achieve this, we double the information content of the xor-bitmask string, and demand that our functions discover this string.

Proof of Theorem 3.2 (sketch): We consider function oracles A of the following form: A is a collection of functions $(f_n)_{n \in \mathbb{N}^+}$ with the following properties:

- (i) $f_n^A : \{0, 1\}^n \rightarrow \{0, 1\}^{n-1}$,
- (ii) f_n^A is 2-to-1,
- (iii) there is a string $s_{n,A}$ in $\{0, 1\}^n - \{0^n\}$ such that for all x of length n , $f_n^A(x \oplus s_{n,A}) = f_n^A(x)$.

Let \mathcal{A} be the set of all such oracles. One can easily induce a probability measure on \mathcal{A} . Indeed, \mathcal{A} is the product of the sets $(\mathcal{A}_i)_{i \in \mathbb{N}^+}$, where, for each $i \in \mathbb{N}^+$, \mathcal{A}_i is the set of all functions f mapping $\{0, 1\}^i$ into $\{0, 1\}^{i-1}$ and having the properties (i), (ii), and (iii). On each set \mathcal{A}_i we consider the probability measure given by the uniform distribution and then we consider the product measure on \mathcal{A} . This is identical to choosing, for each n independently, f_n^A according to the uniform distribution over all functions with the properties (i), (ii), and (iii). All the probabilistic considerations that follow will be relative to this probability measure. It is important to observe that choosing f_n^A uniformly at random amounts to the selection of a random string s of length n and to the independent selection of a random permutation from $\{0, 1\}^{n-1}$ to $\{0, 1\}^{n-1}$ that dictates how the 2^{n-1} pairs $(u, u \oplus s)_{u \in \{0, 1\}^n}$, ordered into some canonical way and identified with $\{0, 1\}^{n-1}$, are mapped into $\{0, 1\}^{n-1}$.

Let $A \in \mathcal{A}$. We define g^A , a function mapping strings of length n into strings of length $2n$, by

$$g^A(w) = s_{2|w|,A},$$

i.e., $g^A(w)$ is the unique string s with the property that for all x of length $2|w|$,

$$f_{2|w|}^A(x \oplus s) = f_{2|w|}^A(x).$$

It follows from the work of Brassard and Høyer [BH97] that there is a machine running in quantum polynomial time that computes g^A for all $A \in \mathcal{A}$.

$\{n = |w|;$
 $a = \text{output of oracle } \mathcal{O} \text{ on input } 0^n;$
 $b = \text{output of oracle } \mathcal{O} \text{ on input } 0^n \oplus w;$
 if $a = b$ then output “ $b(|w|) = 1$ ” else output “ $b(|w|) = 0$.”}

This machine will, on an infinite number of inputs w (on each length n on the input that equals $s(n)$), correctly determine $b(|w|)$ with probability one (relative to the choices of the PTM). Of course, this machine is not correctly accepting $\hat{L} = \{w \mid b(|w|) = 1\}$, but the machine is enough to show that keeping Simon's oracle \mathcal{O} and just adopting the test set \hat{L} does not establish Simon's Theorem 3.5 in the analogous case that applies here, i.e., where any length n string w may be the input.

We will show first that there is a set of oracles \mathcal{B}_0 having measure one in \mathcal{A} , such that for any $A \in \mathcal{B}_0$ any deterministic oracle machine M behaves as follows: for almost every input w , M either runs for more than $2^{|w|/4} - 2$ steps or does not calculate $g^A(w)$.

To move to bounded error probability machines, we invoke the techniques that Bennett and Gill [BG81] used to prove $P^A = BPP^A$ relative to a random oracle. An adaptation of their method shows that there is a set of oracles \mathcal{B}_1 having measure one in \mathcal{A} , such that for any probabilistic oracle machine N , one can build a deterministic oracle machine M with the following property: for any A in \mathcal{B}_1 , if N^A computes a function f with bounded error probability (in the sense of Definition 3.4), then on all inputs w on which $N^A(w)$ runs in time $2^{|w|/5}$, we have that $M^A(w) = f(w)$ and $M^A(w)$ runs for at most $2^{|w|/4} - 2$ steps. If we take $\mathcal{O} \in \mathcal{B}_0 \cap \mathcal{B}_1$ the conclusion of Theorem 3.2 follows. In particular, if for oracle $\mathcal{O} \in \mathcal{B}_0 \cap \mathcal{B}_1$, a machine computes a function in the sense of Definition 3.4 and on infinitely many inputs w runs in $2^{|w|/5}$ steps and has that function coincide with $g^{\mathcal{O}}(w)$, then by the above observation, we would get a deterministic oracle machine M that, for infinitely many w , calculates $g^{\mathcal{O}}(w)$ and runs in $2^{|w|/4} - 2$ steps. But the conclusion of the previous sentence's implication is what we will show is not possible for any oracle in \mathcal{B}_0 .

Thus, let M be a deterministic oracle machine that attempts to calculate g^A . We modify M so that at the end of its computation, having a string s on its output tape, it asks the oracle A for the values of $f_{|s|}^A(0^{|s|})$ and $f_{|s|}^A(s)$. Let M' be the modified machine. The reason for this modification is so we are sure there is a “collision” if M has the correct string s , as we will now make formal and clear. We say that for an oracle A , two strings x and y collide if $f_{|x|}^A(x) = f_{|y|}^A(y)$. Let us fix an input w and let $n = |w|$. Observe that

$$(3.a) \quad \text{Prob}_A(M^A \text{ runs at most } 2^{n/4} - 2 \text{ steps and calculates } g^A(w)) \leq \\ \text{Prob}_A(M'^A \text{ queries at most } 2^{n/4} \text{ strings and} \\ \text{two queried strings of length } 2n \text{ collide with respect to } A),$$

because if M^A is correct on w , then M'^A at the end of its computation will ask 0^{2n} and $s_{2n,A}$ and these will collide.

We assume without loss of generality that for each z and for each oracle A it holds that $M'^A(z)$ does not query the same string twice during its run. Let x_1, x_2, \dots, x_k be, in the order in which they are queried, the strings that M' queries on input w . Of course, k and the set of strings are random variables (in other words they depend on the oracle A). We will show the following fact.

Fact 3.6 $p_w = \text{Prob}_A(k \leq 2^{|w|/4})$ and there is a collision for a pair of strings of length $2|w|$ in $\{x_1, \dots, x_k\} \leq 2^{-1.4|w|}$.

Assuming that the fact holds, we have

$$(3.b) \quad \sum_{w \in \{0,1\}^*} p_w = \sum_{\ell=0}^{\infty} \sum_{w \in \{0,1\}^{\ell}} p_w = \sum_{\ell=0}^{\infty} 2^{\ell} \cdot 2^{-1.4\ell} = \sum_{\ell=0}^{\infty} 2^{-0.4\ell} < \infty.$$

By the Borel-Cantelli Lemma and taking into account (3.a) it follows that

$$\begin{aligned} \text{Prob}_A(\text{for infinitely many inputs } w, M^A(w) \text{ makes at most } 2^{|w|/4} - 2 \text{ queries} \\ \text{and computes } g^A(w) = 0. \end{aligned}$$

Since there are a countable number of deterministic oracle machines M , we obtain that

$$\begin{aligned} \text{Prob}_A(\text{there exists } M \text{ that, on infinitely many inputs } w, \text{ runs at most } 2^{|w|/4} - 2 \text{ steps} \\ \text{and that computes } g^A(w) = 0. \end{aligned}$$

Consequently,

$$\begin{aligned} \text{Prob}_A(\text{for all } M, M^A, \text{ on almost every input } w, \text{ either runs more than } 2^{|w|/4} - 2 \text{ steps} \\ \text{or does not compute } g^A(w) = 1, \end{aligned}$$

which is the desired assertion.

We still must prove Fact 3.6. In this proof, for brevity, collisions will always refer to strings of length $2n$ and will always be with respect to the oracle A . We will drop the subscript from the functions f , with the understanding that the missing subscript is equal to the length of the argument. We will also write $\text{Prob}(\dots)$ for $\text{Prob}_A(\dots)$ when this is clear from the context.

Decomposing the event “ $k \leq 2^{n/4}$ and collision in $\{x_1, \dots, x_k\}$ ” into mutually disjoint events, we have

$$\begin{aligned} (3.c) \quad & \text{Prob}(k \leq 2^{n/4} \text{ and collision in } \{x_1, \dots, x_k\}) = \\ & \text{Prob}(k \leq 2^{n/4} \text{ and collision in } \{x_1, x_2\}) + \\ & \text{Prob}(k \leq 2^{n/4} \text{ and } x_3 \text{ collides with } x_1 \text{ or } x_2 \text{ and no collision in } \{x_1, x_2\}) + \\ & \dots + \\ & \text{Prob}(k \leq 2^{n/4} \text{ and } x_k \text{ collides with } x_1 \text{ or } x_2 \text{ or } \dots \text{ or } x_{k-1} \text{ and no collision in } \{x_1, \dots, x_{k-1}\}) \\ & \leq \sum_{j=2}^{2^{n/4}} \text{Prob}(x_j \text{ collides with } x_1 \text{ or } x_2 \text{ or } \dots \text{ or } x_{j-1} \text{ and no collision in } \{x_1, \dots, x_{j-1}\}), \end{aligned}$$

with the convention that events involving some x_j with $j > k$ are empty (and thus have probability zero). We look at the general term in the above sum.

$$\begin{aligned} (3.d) \quad & \text{Prob}(x_j \text{ collides with } x_1 \text{ or } x_2 \text{ or } \dots \text{ or } x_{j-1} \text{ and no collision in } \{x_1, \dots, x_{j-1}\}) = \\ & \sum \text{Prob}(x_j \text{ collides with } x_1 \text{ or } \dots \text{ or } x_{j-1} \text{ and no collision in } \{x_1, \dots, x_{j-1}\} \mid \\ & (\forall i \in \{1, \dots, j\})[x_i = u_i] \text{ and } (\forall i \in \{1, \dots, j-1\})[f^A(u_i) = a_i]) \times \\ & \text{Prob}((\forall i \in \{1, \dots, j\})[x_i = u_i] \text{ and } (\forall i \in \{1, \dots, j-1\})[f^A(u_i) = a_i]), \end{aligned}$$

where the sum is taken over all j -tuples (u_1, \dots, u_j) of distinct strings in $\{0, 1\}^*$ (that we consider as potential queries of M' on w) and over all possible answers (a_1, \dots, a_{j-1}) to the queries u_1, \dots, u_{j-1} such that the possible answers of length $2n - 1$ are distinct (these are answers to queries of length $2n$ and they are distinct because there is no collision in $\{u_1, \dots, u_{j-1}\}$). Let us fix a tuple (u_1, \dots, u_j) of possible distinct queries and a tuple (a_1, \dots, a_{j-1}) of possible answers as above and let us consider the probability

$$\text{Prob}(x_j \text{ collides with } x_1 \text{ or } \dots \text{ or } x_{j-1} \text{ and no collision in } \{x_1, \dots, x_{j-1}\} \mid \\ (\forall i \in \{1, \dots, j\})[x_i = u_i] \text{ and } (\forall i \in \{1, \dots, j-1\})[f^A(u_i) = a_i]),$$

which is of course equal to

$$(3.e) \quad \text{Prob}(u_j \text{ collides with } u_1 \text{ or } \dots \text{ or } u_{j-1} \text{ and no collision in } \{u_1, \dots, u_{j-1}\} \mid \\ (\forall i \in \{1, \dots, j\})[x_i = u_i] \text{ and } (\forall i \in \{1, \dots, j-1\})[f^A(u_i) = a_i]).$$

Note that the condition “no collision in $\{u_1, \dots, u_{j-1}\}$ ” is subsumed by the condition “ $(\forall i \in \{1, \dots, j-1\})[f^A(u_i) = a_i]$ ” because the answers a_i , for $i = 1, \dots, j-1$, are distinct with respect to those of them of length $2n - 1$. The conditions $f^A(u_i) = a_i$, for $i = 1, \dots, j-1$, completely determine whether it is the case that for all $i \in \{1, \dots, j\}$, the i -th query is u_i , i.e., whether for all $i \in \{1, \dots, j\}$, $x_i = u_i$. Thus the event {no collision in $\{u_1, \dots, u_{j-1}\}$ and $(\forall i \in \{1, \dots, j\})[x_i = u_i]$ and $(\forall i \in \{1, \dots, j-1\})[f^A(u_i) = a_i]$ } is either empty or is equal to the event $\{(\forall i \in \{1, \dots, j-1\})[f^A(u_i) = a_i]\}$. If it is empty, the probability in equation (3.e) is zero (by the standard convention regarding conditional probabilities). In the other case, the probability in equation (3.e) is equal to

$$\text{Prob}(u_j \text{ collides with } \{u_1, \dots, u_{j-1}\} \mid (\forall i \in \{1, \dots, j-1\})[f^A(u_i) = a_i]) \\ = \frac{\text{Prob}(u_j \text{ collides with } \{u_1, \dots, u_{j-1}\} \text{ and } (\forall i \in \{1, \dots, j-1\})[f^A(u_i) = a_i])}{\text{Prob}((\forall i \in \{1, \dots, j-1\})[f^A(u_i) = a_i])}.$$

If $|u_j| \neq 2n$ the above conditional probability is zero. So, we will consider that $|u_j| = 2n$. Let $U = \{u_i \mid i \in \{1, \dots, j-1\} \text{ and } |u_i| = |u_j| = 2n\}$ and let $W = \{u_1, \dots, u_{j-1}\} - U$. Note that $|U|$, the cardinality of U , is at most $j-1$. Observe also that u_j cannot collide with elements from W and that the events “ u_j collides with some element in U and $f^A(u_i) = a_i$, for all u_i in U ” and “ $f^A(u_i) = a_i$, for all u_i in W ” are independent. The events “ $f^A(u_i) = a_i$, for all u_i in U ” and “ $f^A(u_i) = a_i$, for all u_i in W ” are also independent (the choices made in the construction of the oracle at different lengths are independent). Therefore the probabilities involving strings $u \in W$ cancel and it remains to evaluate

$$(3.f) \quad \frac{\text{Prob}(u_j \text{ collides with } \{u_i \mid u_i \in U\} \text{ and } (\forall u_i \in U)[f^A(u_i) = a_i])}{\text{Prob}((\forall u_i \in U)[f^A(u_i) = a_i])}.$$

The events in the above equation depend on the choices of the string s and of the permutation that determines $f_{2n,A}$, and these two choices are independent, as we have observed when

we built the probability measure. Let us focus on the event appearing in the numerator. For this event to hold, the string s , which is responsible for the collisions, must be chosen so as to make u_j collide with one of $\{u_i \mid u_i \in U\}$, and so as to prevent any collision in U (because the “answers” a_i to the “queries” u_i in U are distinct). If we fix one such string s , the 2^{2n-1} pairs $(u, u \oplus s)_{u \in \{0,1\}^{2n}}$ are determined, and the permutation defining A at length $2n$ must be chosen so as to map u_i to a_i for all $u_i \in U$. The number of such permutations does not depend on the fixed string s . Thus, the numerator is equal to the probability over A that $s_{2n,A}$ is in the set $\{u_j \oplus u_i \mid u_i \in U\} - \{u \oplus v \mid u, v \in U \text{ and } u \neq v\}$ times the probability that (for fixed s) the permutation defining A at length $2n$ is compatible with $f^A(u_i) = a_i, u_i \in U$ (a probability that as noted above is the same for each s). The first factor is at most

$$\frac{\|U\|}{2^{2n} - 1} \leq \frac{j - 1}{2^{2n} - 1}.$$

Similarly, the denominator in equation (3.f) is equal to the probability that s is a string of length $2n$ different from 0^{2n} and not in the set $\{u \oplus v \mid u, v \in U \text{ and } u \neq v\}$ times the probability that (for fixed s) the permutation defining A at length $2n$ is compatible with $f^A(u_i) = a_i, u_i \in U$ (and thus the second factor of the denominator is the same as the second factor of the numerator). The first factor of the denominator is

$$\frac{2^{2n} - 1 - \|U\|(\|U\| - 1)/2}{2^{2n} - 1} \geq \frac{2^{2n} - 1 - (j - 1)(j - 2)/2}{2^{2n} - 1}.$$

Consequently, the fraction in equation (3.f) is bounded from above by

$$\frac{j - 1}{2^{2n} - 1 - \frac{(j-1)(j-2)}{2}}.$$

Substituting in equation (3.d), we obtain that

$$\begin{aligned} & \text{Prob}(x_j \text{ collides with } x_1 \text{ or } x_2 \text{ or } \dots \text{ or } x_{j-1} \text{ and no collision in } \{x_1, \dots, x_{j-1}\}) \\ & \leq \frac{j - 1}{2^{2n} - 1 - \frac{(j-1)(j-2)}{2}} \sum \text{Prob}((\forall i \in \{1, \dots, j\})[x_i = u_i] \text{ and } (\forall i \in \{1, \dots, j-1\})[f^A(u_i) = a_i]) \\ & = \frac{j - 1}{2^{2n} - 1 - \frac{(j-1)(j-2)}{2}}. \end{aligned}$$

Thus, returning to equation (3.c), we obtain that

$$\begin{aligned} \text{Prob}(k \leq 2^{n/4} \text{ and collision in } \{x_1, \dots, x_k\}) & \leq \sum_{j=2}^{2^{n/4}} \frac{j - 1}{2^{2n} - 1 - \frac{(j-1)(j-2)}{2}} \\ & \leq \sum_{j=2}^{2^{n/4}} \frac{2^{n/4}}{2^{2n} - 1 - (2^{n/2} - 1)} \\ & \leq \frac{2^{n/2}}{2^{2n} - 2^{n/2}} = \frac{1}{2^{3n/2} - 1} \leq \frac{1}{2^{1.4n}}, \end{aligned}$$

which ends the proof of Fact 3.6. ■

We note that our techniques can be used to show a slightly stronger result: even classical machines that are allowed to err infinitely many times in their computation of the function g still need more than $2^{n/5}$ time on almost every input on which they are correct.

Though it sometimes happens in complexity theory that function results immediately yield corresponding language results, it is not the case that our main result implies, at least in any obvious way, the corresponding language result.⁴

Finally, we mention that in the important (but completely different) area of communication complexity, Raz [Raz99] has shown that for promise problems there is an exponential gap between quantum communication complexity (which in particular is logarithmic on his problem) and classical probabilistic communication complexity (which he gives a lower-bound on as a root of the input size).

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⁴Let us be more explicit, as this point can cause confusion. One might well wonder:

“It seems that your function result will easily give the analogous language result. Why? Basically, by using the standard way we coerce function complexity into language complexity, i.e., via making a language that slices out bits or that prefix searches. For example, using the first of these approaches, take your hard function, call it g . Now consider the function h defined as $h(\langle y, i \rangle) =$ the i ’th bit of $g(y)$. Since g truth-table reduces to h , it follows that if h has fast algorithms then g has fast algorithms (the relation depending on the length of the query strings and the number of queries, but in fact in our case these are such that one could make a good claim). But you prove/claim that g does not have fast classical bounded-error algorithms, so neither can h . And certainly (this actually is the case) Brassard-Høyer easily still gives us that h is quantum-easy to compute.”

However, this reasoning is not valid. The above argument would be fine if we were dealing with infinitely-often hardness. However, we are seeking to prove almost-everywhere hardness, and in fact the bit-slices of an a.e.-hard function are not necessarily a.e.-hard. To see this, consider any a.e.-hard function and prefix a 1 to all its outputs. This is still a.e.-hard but its bit-slices are infinitely often trivial, namely, the first bit of each output is 1. Of course, our hard function does not seem to have any such “obvious” or easy bits, but this is just an informal, tempting hope rather than a valid proof.

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